QUANTITATIVE POLYNOMIAL APPROXIMATION ON CERTAIN PLANAR SETS[1]

BY

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I. Let Q be a compact space in E^k. The *massivity*, $m_n(Q)$, is a sequence defined as follows: Let X_n be a set of $n+1$ elements of Q; then

$$m_n(Q) = \max_{X_n \subset Q} \min_{x_i, x_j \in X_n;\, i \neq j} |x_i - x_j|.$$

Note that m_1 is what is normally defined to be the diameter of the space. Also note that m_n decreases to zero monotonely as $n \to \infty$. E.g., if Q is a set of measure $m > 0$, $m_n(Q)$ is asymptotic to m/n; if Q is a k-dimensional cube, $m_n(Q)$ is asymptotic to $c/n^{1/k}$ [6, p. 21].

Q_1 will represent the interval $0 \leqq x \leqq 1$, and Q_2 the square $0 \leqq x,\ y \leqq 1$. Given a function $f(x)$ defined on Q and a $\delta > 0$, the *modulus of continuity* of the function $f(x)$, $\omega_f(\delta)$, is defined as follows:

$$\omega_f(\delta) = \sup_{|x-y| \leqq \delta} |f(x) - f(y)|.$$

For $f(x)$ in $L^2(Q_1)$, we continue $f(x)$ to have period 1, and we define

$$\omega_f(\delta) = \sup_{|t| \leqq \delta} \left(\int_0^1 |f(x+t) - f(x)|^2 \, dx \right)^{1/2},$$

while for $f(x, y)$ in $L^2(Q_2)$ continued to have period 1 in x and in y

$$\omega_f(\delta) = \sup_{t^2 + s^2 \leqq \delta^2} \left(\int_0^1 \int_0^1 |f(x+t, y+s) - f(x, y)|^2 \, dx \, dy \right)^{1/2}.$$

$\omega(\delta)$ is a nonnegative, nondecreasing continuous function of δ. $f(x)$ is continuous iff $\omega_f(0^+) = 0$. Also, $\omega(\delta)$ is sublinear, and hence $\omega_f(\delta) \geqq c\delta$ for some constant c. If $\omega_f(\delta) \leqq \delta$ for all $\delta > 0$, $f(x)$ is said to be a *contraction* on Q.

Let $C(Q)$ be the space of all real valued continuous functions on Q and let P be a finite dimensional subspace. Let K be the class of contractions on Q. We now introduce the *degree of approximation*

$$\rho_P = \sup_{f \in K} \inf_{p \in P} \|f - p\|.$$

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We have the following *general lower bound theorem* for the degree of approximation: If the dimension of P is n, then $\rho_P \geqq m_n(Q)/2$ [4]. If for some Q there exists c such that $\rho \leqq cm_n(Q)$ for all n, where P is the class of nth degree polynomials, we can then say that P *efficiently* approximates all contractions on Q. If for some family of compact sets Q_ζ there exists c such that $\rho_P(Q) \leqq cm_n(Q)$ for all Q in Q_ζ and for all n, then P can be said to approximate contractions *efficiently on the family* Q_ζ.

The classic upper bound theorem is that of Dunham Jackson which says, essentially, that polynomials approximate efficiently on Q_1 [3, p. 36]. More recently, further results in this direction have been obtained. Yu. A. Brudnji [2] and D. J. Newman simultaneously discovered that for P_n the class of nth degree polynomials on linear sets Q of positive measure, the corresponding degree of approximation, ρ_n, for n sufficiently large, satisfies the inequality $\rho_n \leqq cm/n$, c independent of Q. Newman and H. S. Shapiro have shown that polynomials approximate efficiently on the family of cubes, spheres and balls of all dimensions [6]. On the other hand, polynomials are not efficient on the family of all rectangles in the plane (and, hence, the two-dimensional analog of the result for linear sets of positive measure cannot hold). In fact, there is even a (highly pathological) linear set (of measure zero) on which polynomials are not efficient.

An nth degree polynomial in more than one variable will refer to a polynomial in which the maximum degree in any single variable is less than n. Unless otherwise specified, P and the corresponding ρ will refer to nth degree polynomials. In this case, they will sometimes be denoted, respectively, P_n and ρ_n. (Note that P_n may have dimension greater than n.)

The term *normal curve* will be used for a compact continuous curve of finite length contained in Q_2. In this article we seek estimates for $\rho_P(Q)$ for Q a normal curve.

II. LEMMA 1. *If Q is a normal curve, there exist $c_1, c_2 > 0$ such that $c_1/n \leqq m_n(Q) \leqq c_2/n$.*

Proof. The projection of Q on the x-axis (or the y-axis) must be a line segment $[a, b]$. Consider the points $\{a+k(b-a)/n,\ k=0, 1, \ldots, n\}$. Invert back to $n+1$ corresponding points of Q to get the set X_n. The x-coordinate (or the y-coordinate) of the minimum distance between any two points of X_n is $(b-a)/n$, giving $m_n(Q) \geqq (b-a)/n$. Also, $m_n(Q) \leqq 2L/n$, where L is the length of Q. For, otherwise, there would be $n+1$ nonintersecting discs of radius L/n, with their centers being elements of Q. The portion of Q contained within these discs would have length $\geqq 2L$. This contradiction establishes Lemma 1.

LEMMA 2. *On Q_2, there exist $c_1, c_2 > 0$ such that $c_1/n < \rho_n < c_2/n$.*

This lemma is a special case of the more general theorem of Newman and Shapiro [6, Theorem 4, p. 212].

LEMMA 3. *Let $f(x)$ be a function defined on $S\subset Q$, Q a compact set in E^n, where $f(x)$ has modulus of continuity $\omega(\delta)$. Then there is a function $f^*(x)$ defined on Q with the following properties*:

(a) $f^*(x)=f(x)$, $x\in S$,

(b) $|f^*(x)-f^*(y)|\leqq\omega(d(x,y))$, *where $d(x,y)$ is the distance from x to y. (In other words, a function can be extended without changing its modulus of continuity.)*

Proof. It is sufficient to show that f can be properly extended to one point $\bar{x}$ not in S, for the result would then follow by transfinite induction.

For all x, y in S,

$$\begin{aligned} f(x)-f(y) &\leqq \omega(d(x,y)) \\ &\leqq \omega(d(x,\bar{x})+d(y,\bar{x})), \text{ by the triangle inequality and by the fact that } \omega \text{ is nondecreasing,} \\ &\leqq \omega(d(x,\bar{x}))+\omega(d(y,\bar{x})), \text{ by sublinearity of } \omega. \end{aligned}$$

Hence, for any x we can find α so that

$$f(x)-\omega(d(x,\bar{x})) \leqq \alpha \leqq f(x)+\omega(d(x,\bar{x})).$$

Defining $f^*(\bar{x})$ to be α, we have

$$|f(x)-f(\bar{x})| \leqq \omega(d(x,\bar{x})). \qquad \text{Q.E.D.}$$

COROLLARY. *Let f be a contraction on $S\subset Q$, $Q\subset E^n$. Then f can be extended as a contraction on Q.*

LEMMA 4. *The binomials x^iy^j, $0\leqq i,j\leqq n$, generate on the curve $\sum_{k=0;\,m=0}^{N} a_{km}x^ky^m=0$ a vector space of dimension less than $3Nn$.*

Proof. Let S be the space generated by the x^iy^j, $0\leqq i,j\leqq n$ on the entire plane. Consider the space generated by the x^iy^j, $0\leqq i,j\leqq n$ on the restriction to the given algebraic curve. A basis of the binomials for this space spans, on the plane, a subspace S_0 of S. Another subspace S_1 of S is generated by those binomials spanning S which are not included in the basis for S_0. Note that the dimension of S is $(n+1)^2$. We must show that $\dim S_0<3Nn$. For $n<N$, the result is trivial. Now,

$$x^iy^j\sum_0^N a_{km}x^ky^m, \qquad 0\leqq i,j\leqq n-N,$$

are linearly independent elements of S_1. Hence $\dim S_1\geqq(n-N)^2$. Thus

$$\dim S_0 = \dim S-\dim S_1 \leqq (n+1)^2-(n-N)^2 < 3nN, \qquad n\geqq N,$$

and the lemma is proved.

LEMMA 5. *If Q is a normal curve, then there exist c_1, $c_2>0$ such that $c_1/n^2<\rho_n<c_2/n$ for all n.*

Proof. Since the dimension of the space spanned by nth degree polynomials on Q is $\leqq n^2$, we have $\rho_n \geqq m_{n^2}/2 \geqq c_1/n^2$, by the general lower bound theorem and Lemma 1. To prove the right hand inequality, we can consider any contraction on Q to be extended, by the corollary to Lemma 3, as a contraction to Q_2. Now $\rho_n(Q) \leqq \rho_n(Q_2) \leqq c_2/n^2$, by Lemma 2, completing the proof.

THEOREM 1. *If Q is the restriction of an algebraic curve*

$$\sum_{i=0;\, j=0}^{N} a_{ij} x^i y^j = 0$$

to Q_2, then there exist $c_1, c_2 > 0$ such that $c_1/n \leqq \rho_n(Q) \leqq c_2/n$.

Proof. From Lemma 4, the general lower bound theorem and Lemma 1, $\rho_n \geqq 1/6Nn$, while $\rho_n \leqq c/n$ by Lemma 5.

For Q a normal curve, the trivial estimates of c_1/n and c_2/n^2 as upper and lower bounds for ρ_n have been established. If the dimension of nth degree polynomials on Q is equal to $O(n)$, then $c_1/n < \rho_n < c_2/n$ (as in the case where Q is an algebraic curve). If the dimension of nth degree polynomials on Q is of greater order than n, say of order n^2, to what extent can the trivial estimates for ρ_n be improved? Theorem 2 and Theorem 5 provide a partial answer to this question. It is toward the establishment of these theorems that we conclude this section with several additional lemmata. In Theorem 1 ρ_n can be taken in the L^2 norm or the sup norm, where what is meant by $\|f\|_{L^2}$ on the normal curve $y=g(x)$ is $\left(\int_0^1 |f(x,g(x))|^2\,dx\right)^{1/2}$. In all following results ρ_P will be considered in L^2 only.

We denote by C_P the set of all $\varphi(x)$ with $L^2[0,1]$ norm one which are in the orthogonal complement of P.

LEMMA 6.

$$\rho_P = \sup_{\varphi \in C_P} \left\| \int_0^x \varphi(t)\,dt \right\|_{L^2}.$$

For proof, cf. [5, Lemma 2, p. 942].

We recall that $F(x)$ is said to be in the Paley-Wiener class for the upper half plane, PW, if

(a) F is analytic in the upper half plane, and

$$\int_{-\infty}^{\infty} |F(x+iy)|^2\,dx < M \qquad \text{for all } y>0,$$

or, equivalently,

(b) $$F(z) = \int_0^\infty e^{izx}\varphi(x)\,dx, \qquad \varphi(x) \in L^2[0,\infty].$$

For proof of the equivalence of (a) and (b), and for the general Paley-Wiener theory, the reader is referred to [7, pp. 1–13].

LEMMA 7. *Let* $P=\{x^\lambda,\ \lambda\in\Lambda\}$, *where* $0\in\Lambda$, $\lambda\geqq 0$. *Then*

$$\rho_P^2(Q_1) = \sup_{F\in PW} \frac{\int_{-\infty}^{\infty} \frac{|F(x+i)|^2}{x^2+1/4} \prod_{\lambda\in\Lambda} \frac{x^2+(\lambda-1/2)^2}{x^2+(\lambda+3/2)^2}\,dx}{\int_{-\infty}^{\infty} |F(x)|^2\,dx}.$$

For proof, cf. [5, Lemma 3, p. 943].

It is convenient to adopt the following notation:

$$M(\Lambda) = \max_x \frac{1}{x^2+1/4} \prod_{\lambda\in\Lambda} \frac{x^2+(\lambda-1/2)^2}{x^2+(\lambda+3/2)^2}.$$

LEMMA 8. *For P defined as in Lemma* 7, $\rho_P^2(Q_1)\leqq M(\Lambda)$.

Since, by Parseval's Identity,

$$\int_{-\infty}^{\infty} |F(x+i)|^2\,dx \leqq \int_{-\infty}^{\infty} |F(x)|^2\,dx, \qquad \text{for all } F\in PW,$$

Lemma 8 is a corollary to Lemma 7.

LEMMA 9. *If* $|\sum_{k=0}^{n} a_k x^k|\leqq 1$ *whenever* $0\leqq x\leqq 1$, *then*

$$|a_k| \leqq 2^{2k}\frac{n(n+k-1)!}{(n-k)!\,(2k)!}.$$

Furthermore, these are the best possible bounds for $|a_k|$.

To derive this bound, one demonstrates that $\cos 2n(\arccos x^{1/2})$, which is equal to

$$\sum (-1)^k 2^{2k}\frac{n(n+k-1)!}{(n-k)!\,(2k)!}x^k,$$

is maximal for each coefficient. For proof, see [1, p. 30]. Note that the above upper bounds yield the estimate $|a_k|\leqq 3^{2n}$ for all k.

LEMMA 10. *If* $|\sum_{k=0}^{n^2} a_k x^{k/n}|\leqq 1$, $0<\delta\leqq x\leqq 1$, *then*

$$|a_k| \leqq \left(\frac{3}{1-\delta}\right)^{2n^2}, \qquad k = 0, 1, \ldots, n^2.$$

Proof. Assume $|\sum_0^{n^2} a_k w^k|\leqq 1$, $0<\delta\leqq w\leqq 1$. Let $y=(w-\delta)/(1-\delta)$. Then

$$\left|\sum_0^{n^2} a_k((1-\delta)y+\delta)^k\right| = \left|\sum b_k y^k\right| \leqq 1, \qquad 0\leqq y\leqq 1.$$

Then, by Lemma 9, $|b_k|\leqq 3^{2n^2}$. This, in turn, gives

$$|a_k| < \left(\frac{3}{1-\delta}\right)^{2n^2}.$$

Letting $w=x^{1/n}$, the result follows.

LEMMA 11. *Let p, q, n be integers, $p<q\leqq n$, with p, q relatively prime. The inequality $kq<pm_1+qm_2<(k+1)q$, m_1, m_2, k positive integers, has at least k solutions in m_1, m_2 whenever $k\leqq q-1$. For $q-1<k\leqq n$, the inequality has $q-1$ solutions.*

Proof. The solutions in m_1, m_2 of $kq<pm_1+qm_2<(k+1)q$ are the same as the solutions in m_1, m_2 of $k<m_1p/q+m_2<k+1$. m_1p/q is a nonintegral rational which is less than k for any m_1 in $\{1, 2, \ldots, k\}$. Thus, letting $m_2=1+[k-m_1p/q]$, there is exactly one solution for $m_1=1, 2, \ldots, k$, hence at least k solutions.

Similarly, for $k\geqq q$, $m_1p/q+1+[k-m_1p/q]$ are solutions for $m=1, 2, \ldots, q-1$, and the lemma is proved.

III. Our main theorems are Theorem 2 and Theorem 5.

THEOREM 2. *If Q is the set (x, e^x), $0\leqq x\leqq 1$, then there exist c_1, $c_2>0$ such that*

$$\frac{c_1}{n^{3/2}} \leqq \max_{f\in k} \min_{p\in P_n} \|f(x, e^x)-p(x, e^x)\|_{L^2} \leqq \frac{c_2}{n^{3/2}},$$

i.e., $c_1/n^{3/2}\leqq \rho_n\leqq c_2/n^{3/2}$ in the L^2 norm.

Let α be an irrational number. α will be called *of degree $f(n)$* if there exist p, n such that $|\alpha-p/n|<1/f(n)$ for infinitely many n.

THEOREM 5. *Let α, $0<\varepsilon\leqq\alpha\leqq 1$ be an irrational of degree $n^4(3/(1-\delta))^{2n^2}$, $0<\delta<1$. Let Q be the set of points (x, x^α), $\delta\leqq x\leqq 1$. Then there exists a subsequence ρ_{n_i} of ρ_n and constants c_1, $c_2>0$ such that, in the L^2 norm, $c_1/n_i^{3/2}\leqq\rho_{n_i}\leqq c_2/n_i^{3/2}$.*

Proof of Theorem 2. We are approximating with linear combinations of x^ke^{mx}, $k, m\leqq n$. By Lemma 6,

$$\rho_n = \sup_{\varphi\in C_P}\left\|\int_0^x \varphi(t)\,dt\right\|, \quad \text{where } \int_0^1 \varphi(x)x^ke^{mx}\,dx = 0, \quad k, m \leqq n,$$

or, letting $te=e^x$, where

$$\int_{1/e}^1 \varphi(t)(\log t)^kt^{m-1}\,dt = 0, \qquad k, m \leqq n. \tag{1}$$

Let

$$F(z) = \int_{1/e}^1 \varphi(t)t^{-(iz+1/2)}\,dt.$$

Letting $t=e^{-u}$, we note that $F\in PW$. Setting $k=0$ in (1), we get $F((m+\frac{1}{2})i)=0$, $m\leqq n-1$. Setting $k=1$ in (1) and integrating by parts, we get $F'((m+\frac{1}{2})i)=0$, $m\leqq n-1$. For general k, integration by parts k times yields

$$F^{(k)}((m+\tfrac{1}{2})i) = 0, \qquad m \leqq n-1. \tag{2}$$

(In each case the boundary terms drop out by the orthogonality of φ to x^ke^{mx}.)

Since $F(x)=\int_{-\infty}^{\infty} e^{iux}e^{-u/2}\varphi(e^{-u})\,du$, Parseval's Identity yields

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(x)|^2\,dx = \int_{-\infty}^{\infty} e^{-u}|\varphi(e^{-u})|^2\,du = \int_0^1 |\varphi(t)|^2\,dt = 1. \tag{3}$$

Integrating by parts,

$$F(z) = t^{-(iz+1/2)}\int_0^t \varphi(u)\,du + (iz+1/2)\int_{1/e}^1 t^{-(iz+3/2)}\int_0^t \varphi(u)\,du\,dt.$$

But φ is orthogonal to 1, hence

$$F(z) = (iz+1/2)\int_{1/e}^1 t^{-(iz+3/2)}\int_0^t \varphi(u)\,du\,dt.$$

Thus

$$\frac{F(x+i)}{ix+1/2} = \int_{1/e}^1 t^{-(ix+1/2)}\int_0^t \varphi(u)\,du\,dt$$
$$= \int_{-\infty}^{\infty} e^{ivx-v/2}\int_0^{e^{-v}} \varphi(u)\,du\,dv.$$

Hence, by Parseval's Identity,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|F(x+i)|^2}{x^2+1/4}\,dx = \int_{-\infty}^{\infty} e^{-v}\left|\int_0^{e^{-v}}\varphi(u)\,du\right|^2 dv = \int_0^1\left|\int_0^t \varphi(u)\,du\right|^2 dt. \tag{4}$$

It follows from (2) and the general Paley-Wiener theory that

$$G(z) = F(z)\prod_{m\leqq n-1}\left(\frac{z+i(m+1/2)}{z-i(m+1/2)}\right)^n$$

is in PW. Further, since $|x+i(m+1/2)|=|x-i(m+1/2)|$, (3) gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|G(x)|^2\,dx = 1, \tag{5}$$

and (4) becomes

$$\int_0^1\left|\int_0^t\varphi(u)\,du\right|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|G(x+i)|^2}{x^2+1/4}\prod_{m\leqq n-1}\left(\frac{x^2+(m-1/2)^2}{x^2+(m+3/2)^2}\right)^n dx. \tag{6}$$

From (5), (6) and Lemma 6 we get

$$\rho_n^2 = \sup_{G\in PW}\frac{\displaystyle\int_{-\infty}^{\infty}\frac{|G(x+i)|^2}{x^2+1/4}\prod_{m\leqq n-1}\left(\frac{x^2+(m-1/2)^2}{x^2+(m+3/2)^2}\right)^n dx}{\displaystyle\int_{-\infty}^{\infty}|G(x)|^2\,dx}. \tag{7}$$

But $\|G(x+i)\|_{L^2}^2 \leqq \|G(x)\|_{L^2}^2$. Hence

$$\rho_n^2 \leqq \max_x \frac{1}{x^2+1/4}\prod_{m\leqq n-1}\left(\frac{x^2+(m-1/2)^2}{x^2+(m+3/2)^2}\right)^n$$
$$= \max_x \frac{(x^2+1/4)^{2n-1}}{(x^2+(n-1/2)^2)^n(x^2+(n+3/2)^2)^n}.$$

We note that the function $x=x(n)$ at which the max is taken on must go to infinity with n, for, if not, the upper bound thus obtained for ρ_n would grossly violate the general lower bound theorem. This being the case,

$$\rho_n < c \max_x \frac{x^{2n-1}}{(x^2+n^2)^n}.$$

The max occurs at $x^2=2n^3-n^2$, and the maximum value is $c(2n^3-n^2)^{-1/2}(1-1/2n)^n$, which tends to $c/(2e)^{1/2}n^{3/2}$, giving the desired upper bound.

Choosing $G(z)=1/(z+in^{3/2})$ we get from (7) that

$$\rho_n^2 \geqq \frac{\int_{-\infty}^{\infty} \frac{c}{x^2}\left(\frac{x^2}{x^2+n^2}\right)^{2n} \frac{dx}{x^2+n^3}}{\int_{-\infty}^{\infty} \frac{dx}{x^2+n^3}}$$

$$\geqq \frac{n^{3/2}}{\pi} \int_{n^{3/2}}^{2n^{3/2}} \frac{c'}{x^2}\left(\frac{x^2}{x^2+n^2}\right)^{2n} \frac{dx}{x^2+n^3}$$

$$\geqq \frac{c'}{\pi 2e^2n^3} \int_1^2 \frac{du}{u^2+1} = \frac{c''}{n^3}. \qquad \text{Q.E.D.}$$

Our interest will now, temporarily, be turned from linear combinations of x^k, $k=0, 1, \ldots, n$ as approximating functions to approximation with linear combinations of $x^{k/q}$, $k=0, 1, 2, \ldots, nq$, on the interval Q_1. We refer to the latter space and the corresponding degree of approximation as P_{nq}^* and ρ_{nq}^*.

Since $P_n \subset P_{nq}^*$, it follows from Jackson's theorem that there is a c_1 such that $\rho_{nq}^* \leqq c_1/nq$. By the general lower bound theorem, there is a c_2 such that $\rho_{nq}^* \geqq c_2/n$. The following result, in addition to being of independent interest, will be used in the proof of Theorem 4.

THEOREM 3. *There exist constants c_1, $c_2>0$ such that $c_1/nq^{1/2} \leqq \rho_{nq}^* \leqq c_2/nq^{1/2}$.*

Proof. By Lemma 7,

$$\rho_{nq}^{*2} = \sup_{F\in PW} \frac{\int_{-\infty}^{\infty} \frac{|F(x+i)|^2}{x^2+1/4} \prod_{k=0}^{nq} \frac{x^2+(k/q-1/2)^2}{x^2+(k/q+3/2)^2}\, dx}{\int_{-\infty}^{\infty} |F(x)|^2\, dx}, \tag{8}$$

while, by Lemma 8,

$$\rho_{nq}^{*2} \leqq M(\Lambda), \quad \text{where } \Lambda = \{k/q \mid k = 0, 1, \ldots, nq\}. \tag{9}$$

But, by cancellation,

$$M(\Lambda) = \max_x\, (x^2+1/4)^{-1} \prod_{k=0}^{2q-1} (x^2+(k/q-1/2)^2) \prod_{j=qn-2q+1} (x^2+(j/q+3/2)^2)^{-1}. \tag{10}$$

By (9) and by the general lower bound theorem, M cannot fall off faster than c/n^4. The point $x=x(n, q)$ where the max is taken on must go to infinity with n or q, for, if not, (10) would yield $M=O(1/n^{2qn})$. We now have, from (10), that there exist positive constants k_1, k_2 such that

$$k_1 M \leqq \frac{1}{x^2}\left(\frac{x^2}{x^2+n^2}\right)^{2q} \leqq k_2 M. \tag{11}$$

The max in (11) is determined to be at $x^2=2qn^2-n^2$, yielding $M=O(1/n^2q)$. Thus, from (9) we have

$$\rho^*_{nq} = O(1/nq^{1/2}). \tag{12}$$

Choosing $F(z)=1/(z+inq^{1/2})$, we have, from (8) and (11),

$$\rho^{*2}_{nq} \geqq \frac{\int_{-\infty}^{\infty} \frac{c}{x^2}\left(\frac{x^2}{x^2+n^2}\right)^{2q} \frac{dx}{x^2+n^2q}}{\int_{-\infty}^{\infty} \frac{dx}{x^2+n^2q}}$$

$$\geqq \frac{nq^{1/2}}{\pi}\int_{nq^{1/2}}^{2nq^{1/2}} \frac{c}{x^2}\left(\frac{x^2}{x^2+n^2}\right)^{2q} \frac{dx}{x^2+n^2q}$$

$$\geqq \frac{c}{\pi nq^{1/2}}(1+1/q)^{-2q}\int \frac{dx}{x^2+n^2q} \geqq \frac{c'}{n^2q},$$

which, together with (12), proves the theorem.

Let Q be the curve $y=x^{p/q}$, p, q relatively prime, $\delta \leqq x \leqq 1$. Consider the family of all such Q. Since these curves are algebraic, Theorem 1 implies $c_1/n \leqq \rho_n \leqq c_2/n$. However, for each Q in the family, the constants, c_1, c_2, although independent of n, might well depend on which particular curve from the family is under consideration, i.e, c_1 and/or c_2 may be nontrivial functions of p and/or q. We shall now concern ourselves with more precise estimates for ρ_n—estimates that will yield the order of magnitude as a function of which Q from the family is under consideration. The following result in this direction will be crucial to the proof of Theorem 5.

THEOREM 4. *Let Q be the set $(x, x^{p/q})$, $\delta \leqq x \leqq 1$, p, q relatively prime. Then there exist c_1, $c_2>0$ such that, in the L^2 norm, $c_1/nq^{1/2} \leqq \rho_n \leqq c_2/nq^{1/2}$ for n, p, q such that $n \geqq q$.*

Proof. We note that for $y=x^{p/q}$, approximating with nth degree polynomials in x and y is the same as approximating with $x^{i+jp/q}$, $0 \leqq i, j \leqq n$.

Let $f^*(x)=f(x, x^{p/q})$ be defined for $\delta \leqq x \leqq 1$. Now

$$\|f(x, x^{p/q})-p_n(x, x^{p/q})\| = \left\|f^*(x)-\sum a_{ij}x^{i+jp/q}\right\|.$$

Also, the slope of $f^*(x)$,

$$\frac{\|f^*(y)-f^*(x)\|}{d(x, y)} = \frac{\|f(y, y^{p/q})-f(x, x^{p/q})\|}{d((x, x^{p/q}), (y, y^{p/q}))}\cdot\frac{d((x, x^{p/q}), (y, y^{p/q}))}{d(x, y)} \leqq (1+m^2)^{1/2},$$

where m is the maximum slope for $y=x^{p/q}$ on $[\delta, 1]$. Since $m<1/\delta$, we have

$\|f^*(y)-f^*(x)\| \leqq M|y-x|$, $M=(1+1/\delta^2)^{1/2}$. Thus $f^*(x)/M$ is in $K(\delta, 1)$. Hence the assertion of our theorem is equivalent to the existence of c_1, c_2 such that

$$\frac{c_1}{nq^{1/2}} \leqq \max_{f\in K(\delta,1)} \min_{\{a_{ij}\}} \left\| f(x) - \sum_{i,j=0}^{n} a_{ij}x^{i+jp/q} \right\| \leqq \frac{c_2}{nq^{1/2}}. \tag{13}$$

First, the left inequality in (13) will be established. Define

$$\rho^*_{nq}(a, b) \max_{f\in K(a,b)} \min_{\{a_k\}} \left\| f(x) - \sum_{k=0}^{nq} a_k x^{k/q} \right\|.$$

LEMMA 12. $\rho^*_{2nq}(\delta, 1) \geqq (1-\delta)\rho^*_{2nq}(0, 1)$.

Proof. With respect to $\rho^*_{2nq}(0, 1)$, consider the coefficients $\{b_k\}$ for the function, g, that maximizes. Then

$$\left\| g(x) - \sum_0^{2nq} b_k x^{k/q} \right\| = \rho^*_{2nq}(0, 1).$$

Let $f(x) = (1-\delta)g((x-\delta)/(1-\delta))$. $f(x)$ is in $K(\delta, 1)$. Now

$$\sum_0^{2nq} a_k x^{k/q} = \sum (1-\delta)b_k((x-\delta)/(1-\delta))^{k/q}$$

is the best approximating polynomial to $f(x)$ on $[\delta, 1]$. For, suppose the contrary, i.e.,

$$\left\| \sum c_k x^{k/q} - f(x) \right\| < \left\| \sum (1-\delta)b_k((x-\delta)/(1-\delta))^{k/q} - f(x) \right\|$$

on $[\delta, 1]$. Then, on $[0, 1]$,

$$\begin{aligned} \left\| \sum c_k((1-\delta)x+\delta)^{k/q} - f((1-\delta)x+\delta) \right\| &\leqq \left\| \sum (1-\delta)b_k x^{k/q} - f((1-\delta)x+\delta) \right\| \\ &\leqq \left\| \sum (1-\delta)b_k x^{k/q} - g(x) \right\|, \end{aligned}$$

contrary to the assumption that the b_k minimized for g. Hence,

$$\begin{aligned} \rho^*_{2nq}(\delta, 1) &\geqq \left\| \sum (1-\delta)b_k((x-\delta)/(1-\delta))^{k/q} - f(x) \right\| \quad \text{on } [\delta, 1] \\ &= (1-\delta)\left\| \sum b_k((x-\delta)/(1-\delta))^{k/q} - g((x-\delta)/(1-\delta)) \right\| \quad \text{on } [\delta, 1] \\ &= (1-\delta)\left\| \sum b_k x^{k/q} - g(x) \right\| \quad \text{on } [0, 1] \\ &= (1-\delta)\rho^*_{2nq}(0, 1), \end{aligned}$$

and Lemma 12 is established. Now,

$$\max_{f\in K(\delta,1)} \min_{\{a_{ij}\}} \left\| f(x) - \sum a_{ij}x^{i+jp/q} \right\| \geqq \rho^*_{2nq}(\delta, 1) \geqq (1-\delta)\rho^*_{2nq}(0, 1) \geqq \frac{c}{nq^{1/2}},$$

the last inequality following from Theorem 3.

We now prove the upper bound in (13). Let f be extended, as a contraction, to $[0, 1]$. Lemma 8 now reduces the proof of the desired inequality to showing that

$$M(\Lambda) < c/n^2q, \quad \text{where } \Lambda = \{i+jp/q,\ 0 \leqq i, j \leqq n\}.$$

Let $x=x(n,q)$ be the point x at which the max in $M(\Lambda)$ is taken on. As in the proof of Theorem 2, $x(n,q)$, for fixed q, must tend to infinity with n. Observe that $(x^2+(\lambda-1/2)^2)/(x^2+(\lambda+3/2)^2)$ is an increasing function of λ for $\lambda>4(1+x^2)^{1/2}-2$, and decreasing for λ less than this quantity. Let q be fixed. We consider two possible cases:

Case I. $1/x(n,q)=O(1/n)$.

Then there is some c such that $(x^2+(\lambda-1/2)^2)/(x^2+(\lambda+3/2)^2)$ is decreasing for all $\lambda<cn$. Let Λ_0 be $\{\lambda\in\Lambda \mid \lambda<cn\}$. $M(\Lambda_0)\geqq M(\Lambda)$. The facts that

$$(x^2+(\lambda-1/2)^2)/(x^2+(\lambda+3/2)^2)<1,$$

that this quotient is a decreasing function of λ, and that, by Lemma 11, there are at least $\min(k,q-1)$ members of Λ_0 between k and $k+1$ for $k+1<cn$ imply that $M(\Lambda_1)>M(\Lambda_0)$, where

$$\Lambda_1=\{i+j/q,\ 0\leqq i\leqq [cn],\ 0\leqq j\leqq \min(i,q)\}.$$

That is,

$$\begin{array}{llllll}
\Lambda_1=\{0, & & & & & \\
1, & 1+1/q, & & & & \\
2, & 2+1/q, & 2+2/q, & & & \\
3, & 3+1/q, & 3+2/q, & 3+3/q, & & \\
4, & 4+1/q, & 4+2/q, & 4+3/q, & & \\
\vdots & \vdots & \vdots & \vdots & & \\
q-1, & q-1+1/q, & q-1+2/q, & & \ldots, & q-1+(q-1)/q, \\
q, & q+1/q, & q+2/q, & & \ldots, & q+(q-1)/q, \\
\vdots & \vdots & \vdots & & & \vdots \\
[cn-1], & [cn-1]+1/q, & [cn-1]+2/q, & & \ldots, & [cn-1]+(q-1)/q\}.
\end{array}$$

By the cancellation of the terms in the product (all but the first two and last two entries in each column cancel) we have

$$M(\Lambda_1)=\max_x \frac{1}{x^2+\frac{1}{4}}\prod_{k=0}^{q}\frac{x^2+(k+k/q-1/2)^2}{x^2+([cn]+k/q+1/2)^2}\cdot\frac{x^2+(k+k/q+1/2)^2}{x^2+([cn]+k/q+3/2)^2}$$

which is of the order of

$$\max_x \frac{1}{x^2}\left(\prod\left(1-\frac{c^2n^2-k^2}{x^2+c^2n^2}\right)\right)^2<\max_x\frac{1}{x^2}\exp\left(-2\sum^{q}\frac{c^2n^2-k^2}{x^2+c^2n^2}\right),$$

or

$$\max_x\frac{1}{x^2}\exp\left(-\frac{2c^2n^2q-q^3/3}{x^2+n^2}\right).$$

But the maximum (away from the origin) is taken at $x^2=n^2(2cq^2-1)-q^3/3$, where we get $M(\Lambda_1)=O(1/qn^2)$, and the estimate is established for Case I.

Case II. We now suppose that $1/x(n,q)$ is not $=O(1/n)$. Then there is a subsequence of $\{n\}$ for which $x(n,q)=O(n)$. Let $\Lambda_2=\{\lambda\in\Lambda \mid \lambda>[4x]\}$. $M(\Lambda_2)>M(\Lambda)$. Now, that $(x^2+(\lambda-1/2)^2)/(x^2+(\lambda+3/2)^2)$ is an increasing function of λ bounded by 1 implies, in view of Lemma 11, that $M(\Lambda_3)>M(\Lambda_2)$, where

$$\Lambda_3 = \{i-j/q \mid [4x] \leqq i \leqq n,\ 0 \leqq j \leqq \min(i,q)\}.$$

That is,

$$\begin{aligned}\Lambda_3 = \{&[4x]-\min([4x],q-1)/q, && \ldots, && [4x]-2/q, && [4x]-1/q, && [4x],\\ &[4x]+1-\min([4x]+1,q-1)/q, && \ldots, && [4x]+1-2/q, && [4x]+1-1/q, && [4x]+1,\\ &[4x]+2-\min([4x]+2,q-1)/q, && \ldots, && [4x]+2-2/q, && [4x]+2-1/q, && [4x]+2,\\ &\vdots && && \vdots && \vdots && \vdots\\ &n-(q-1)/q, && \ldots, && n-2/q, && n-1/q, && n\}.\end{aligned}$$

By cancellation of terms in the product, we have

$$M(\Lambda_3) \leqq \max_x \frac{1}{x^2+\frac{1}{4}} \prod_{k=0}^{\min([4x,q])} \frac{x^2+([4x]-k/q-1/2)^2}{x^2+(n-k/q+1/2)^2}\cdot\frac{x^2+([4x]-k/q-3/2)^2}{x^2+(n-k/q-1/2)^2}.$$

Thus, for the subsequence of n for which $x=O(n)$, we have

$$M(\Lambda_3) = O(cx^2/n^2)^\alpha, \quad \alpha = \min([4x],q).$$

Hence, for these values of n, $M(\Lambda)<M(\Lambda_3)<1/n^5$, which is impossible by the lower bound theorem. Thus Case II cannot occur and the theorem is established.

We are now in a position where we can give a proof of Theorem 5: Consider all n for which there exist p such that $|\alpha-p/n|<1/n^2(3/(1-\delta))^{2n^2}$. This is a subsequence n_i of n, and defines our subsequence ρ_{n_i} of ρ_n. For convenience of notation, the subscripts will be dropped, i.e., n_i will be referred to as n. It will be shown that

$$\varepsilon c_1/2n^{3/2} \leqq \rho_n \leqq 2c_2/\varepsilon n^{3/2}. \tag{14}$$

(Here c_1 and c_2 are the same constants as those in Theorem 4.)

Extend $f(x,y)$, now defined as a contraction on $y=x^\alpha$, to be a contraction on the square $0\leqq x$, $y\leqq 1$. That this can be done follows, again, from the corollary to Lemma 3. Consider the curve $y=x^{p/n}$. Then

$$|x^\alpha - x^{p/n}| < 1/\varepsilon n^4(3/(1-\delta))^{2n^2}. \tag{15}$$

By Theorem 4, there exists $p_n(x,y)$ such that

$$\|p_n(x,x^{p/n})-f(x,x^{p/n})\| \leqq c_2/n^{3/2}. \tag{16}$$

Now,

$$\begin{aligned}\|p_n(x,x^\alpha)-f(x,x^\alpha)\| \leqq\ & \|p_n(x,x^\alpha)-p_n(x,x^{p/n})\|\\ &+\|p_n(x,x^{p/n})-f(x,x^{p/n})\|\\ &+\|f(x,x^{p/n})-f(x,x^\alpha)\|.\end{aligned} \tag{17}$$

Since f is a contraction, it follows from (15) that

$$\|f(x, x^{p/n})-f(x, x^{\alpha})\| \leqq 1/\varepsilon n^4(3/(1-\delta))^{2n^2}. \tag{18}$$

Now,

$$\|p_n(x, x^{\alpha})-p_n(x, x^{p/n})\| \leqq \max_{\delta\leqq x\leqq 1} |p_n(x, x^{\alpha})-p_n(x, x^{p/n})|$$

$$\leqq \max_{\delta\leqq x\leqq 1} \sum^{n} \left|\left(\frac{d^k}{dy^k} p_n(x, y)\right)_{y=x^{p/n}}\right| |x^{\alpha}-x^{p/n}|^k,$$

by Taylor's Theorem. But this is

$$< \frac{1}{\varepsilon n^4(3/(1-\delta))^{2n^2}} \sum \left|\frac{d^k}{dy^k} p_n(x, y)\right|_{y=x^{p/n}}$$

by (15). But, by Lemma 10, the coefficients of $p_n(x, x^{p/n})$ are bounded by $(3/(1-\delta))^{2n^2}$. Hence, we have

$$\|p_n(x, x^{\alpha})-p_n(x, x^{p/n})\| < 1/\varepsilon n^2. \tag{19}$$

Substitution of (16), (18), and (19) in (17) yields the upper bound asserted in (14).

To establish the lower bound, we note that

$$\|p_n(x, x^{\alpha})-f(x, x^{\alpha})\| + \|p_n(x, x^{\alpha})-p_n(x, x^{p/n})\| + \|f(x, x^{p/n})-f(x, x^{\alpha})\| \geqq \|p_n(x, x^{p/n})-f(x, x^{p/n})\|,$$

and hence that by (18), (19) and Theorem 4,

$$\|p_n(x, x^{\alpha})-f(x, x^{\alpha})\| \geqq \frac{c_1}{n^{3/2}}-\frac{1}{n^4}-\frac{1}{n^2} \geqq \frac{c_1}{2n^{3/2}}. \qquad \text{Q.E.D.}$$

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